

## Distribution of reflection eigenvalues in many-channel chaotic cavities with absorption

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(Received 12 November 2003; published 11 March 2004)

The reflection matrix  $R = S^\dagger S$ , with  $S$  being the scattering matrix, differs from the unit matrix when absorption is finite. Using the random matrix approach, we calculate analytically the distribution function of its eigenvalues in the limit of a large number of propagating modes in the leads attached to a chaotic cavity. The obtained result is independent of the presence of time-reversal symmetry in the system, being valid at finite absorption and arbitrary openness of the system. The particular cases of perfectly and weakly open cavities are considered in detail. An application of our results to the problem of thermal emission from random media is briefly discussed.

DOI: 10.1103/PhysRevE.69.035201

PACS number(s): 05.45.Mt, 42.25.Bs, 73.23.-b, 42.50.Ar

### INTRODUCTION

When absorption in a cavity is finite, part of the incoming flux gets irreversibly lost in the walls, thus breaking the unitarity of the scattering matrix. The mismatch between incoming and outgoing fluxes in the scattering process can be naturally described by means of the reflection matrix  $R = S_\gamma^\dagger S_\gamma$ , where  $S_\gamma$  denotes the subunitary (at nonzero absorption) scattering matrix to be precisely defined below. The matrix  $R$  has the positive eigenvalues  $r_c \leq 1$ , the so-called *reflection eigenvalues*. Their statistical properties in chaotic resonance scattering attract much attention presently, both experimentally [1,2] and theoretically [3–9], using the random-matrix theory approach [10,11].

In the limit of weak absorption,  $R$  was found [1,7] to be related to the Smith's time-delay matrix at zero absorption, whose statistical properties were extensively studied in recent time [12–16]. Such an analysis was recently generalized by us to the case of arbitrary absorption [8], see also [9] for the related study. In the particular case of the single-channel cavity, obtained results partly explain the recent experiment [2] on the reflection coefficient distribution.

The opposite case of a large number of equivalent channels plays an important role in chaotic scattering, corresponding to the semiclassical limit of matrix models [17,18]. In this case  $N$  strongly overlapping resonances (poles of the  $S$  matrix) are excited in the open cavity through the scattering channels, whose number  $M$  scales with  $N$ . A new energy scale appears then in addition to the mean level spacing  $\Delta$  of the closed cavity [12,18]: the empty gap between the real axis in the complex energy plane and the cloud of the resonances in its lower half plane. In the physically justified limit  $M/N \ll 1$  [12] (though both  $M, N \rightarrow \infty$ ) considered below this gap is given by the well-known Weisskopf width  $\Gamma_w = MT\Delta/2\pi$ , with  $T = 1 - |\bar{S}|^2 \leq 1$  being the transmission coefficient [17]. In the presence of finite absorption the widths of the resonances acquire an additional contribution, the absorption width  $\Gamma_a$  [8]. The two competing processes, the escape into continuum and dissolution into the walls, are therefore characterized by the ratio  $\Gamma_a/\Gamma_w = \gamma/T$ , where the convenient dimensionless parameter  $\gamma \equiv 2\pi\Gamma_a/\Delta M$  measures the absorption strength.

In this Rapid Communication we derive the probability distribution function  $\mathcal{P}(r) = M^{-1} \overline{\sum_{c=1}^M \delta(r - r_c)}$  of reflection eigenvalues, with the bar indicating the statistical average, in the limit of the large number of channels. This function has, in particular, an important application for the statistics of thermal emission from random media and is known only at perfect coupling ( $T = 1$ ) [5]. Here, we calculate it at arbitrary  $T$ .

Our starting point is the general relation established in Ref. [8] between  $R$  and the effective non-Hermitian Hamiltonian  $\mathcal{H} = H - (i/2)VV^\dagger$  of the open system, where the Hermitian part  $H$  stands for the closed counterpart and the amplitudes  $V_n^c$  describe the coupling between  $N$  intrinsic and  $M$  channel states. It is achieved by making use of the equivalence [6–8] of uniform absorption to the pure imaginary shift  $E_\gamma \equiv E + i\Gamma_a/2$  of the scattering energy  $E$ , thus giving  $S_\gamma \equiv S(E_\gamma) = 1 - iV^\dagger(E_\gamma - \mathcal{H})^{-1}V$ . This leads to the following connection between the reflection matrix  $R$  and the time-delay matrix with absorption  $Q_\gamma$  ( $\hbar = 1$ ) [8]:

$$R = 1 - \Gamma_a Q_\gamma, \quad (1)$$

where  $Q_\gamma \equiv Q(E_\gamma) = V^\dagger[(E_\gamma - \mathcal{H})^\dagger]^{-1}(E_\gamma - \mathcal{H})^{-1}V$ . The distribution function  $\mathcal{P}(r)$  is therefore directly related as

$$\mathcal{P}(r) = \gamma^{-1} \mathcal{P}_\tau[\gamma^{-1}(1-r)] \quad (2)$$

to the distribution  $\mathcal{P}_\tau(\tau) = M^{-1} \overline{\sum_{c=1}^M \delta(\tau - Mq_c/t_H)}$  of the proper delay times (eigenvalues  $q_c$  of  $Q_\gamma$ ) scaled with  $M$  and measured in units of the Heisenberg time  $t_H = 2\pi/\Delta$ .

The function (2) is normalized to unity. The average reflection coefficient  $\langle r \rangle = M^{-1} \overline{\text{tr} R}$  is given by

$$\langle r \rangle = \int_0^1 dr r \mathcal{P}(r) = \frac{T + \gamma(1-T)}{T + \gamma}. \quad (3)$$

This result follows from the connection [8] between  $\langle r \rangle$  and the ‘‘norm-leakage’’ decay function  $P(t)$  introduced in [19]:  $\langle r \rangle = 1 - \gamma [1 - \Gamma_a \int_0^\infty dt e^{-\Gamma_a t} P(t)]$ . In the limit considered  $P(t)$  reduces to the simple exponent  $e^{-\Gamma_w t}$ , since the widths ceased to fluctuate [18,19]. Below we will also derive Eq. (3) using a different method.

### THE SADDLE-POINT EQUATION

The scaling limit studied is far from being trivial and cannot be obtained from the known expressions for  $\mathcal{P}(r)$  at finite  $M$  [8] simply by letting  $M \rightarrow \infty$  there. One needs to start from the original definition that determines the desired distribution through the jump of the resolvent  $G(t) = M^{-1} \text{tr}(t - \Gamma_a Q_\gamma)^{-1} \equiv 1/t + (\gamma/t^2)K(t)$  along the discontinuity line  $t > 0$  as follows:

$$\mathcal{P}(r) = \frac{\gamma}{\pi(1-r)^2} \text{Im} K(t-i0)|_{t=1-r}. \quad (4)$$

Following Refs. [8,16], one can obtain the exact (at  $N \rightarrow \infty$ ) representation for  $K$ , which is convenient for our purposes,

$$K(t) = \left\langle \frac{1}{8} \text{str} \left[ \left( \Lambda - i \frac{1-t/2}{\sqrt{1-t}} \Lambda_1 \right) k \hat{\sigma} \right] \right\rangle_{\mathcal{L}}, \quad (5)$$

with the shorthand  $\langle (\dots) \rangle_{\mathcal{L}} = \int d\mu(\hat{\sigma}) (\dots) e^{(M/2)\mathcal{L}}$  and

$$\begin{aligned} \mathcal{L} = & (\gamma/t) \text{str}[(1-t/2)\Lambda \hat{\sigma} - i\sqrt{1-t}\Lambda_1 \hat{\sigma}] \\ & - \text{str} \ln[1 + (2g)^{-1}(\Lambda \hat{\sigma} + \hat{\sigma} \Lambda)], \end{aligned} \quad (6)$$

as the integral over the noncompact saddle-point manifold of the  $8 \times 8$  supermatrix  $\hat{\sigma}$  subject to the constraint  $\hat{\sigma}^2 = 1$ .  $\Lambda, \Lambda_1$  appearing above are the supermatrix analog of the Pauli matrices  $\sigma_3, \sigma_1$  and  $k = +1(-1)$  in the space of commuting (anticommuting) variables. Definitions of the superalgebra as well as explicit parametrization, which depends on whether time-reversal symmetry (TRS) is preserved or broken, can be found in [10,20]. At last, the constant  $g = 2/T - 1 \geq 1$  is related to the transmission coefficient  $T$ .

In the limit  $M \rightarrow \infty$  the integration over  $\hat{\sigma}$  can also be done in the saddle-point approximation. Function (5) is given by a saddle-point value of  $\hat{\sigma}$ , which is found, as usual, by equating the variance  $\delta\mathcal{L}$  to zero. The structure of the latter equation is determined by that of  $\mathcal{L}$ . A careful analysis shows that, independently of TRS, the supermatrix  $\hat{\sigma}$  reduces (and so does the corresponding algebra) in the saddle-point to the  $2 \times 2$  usual matrix  $\sigma = a\sigma_3 + b\sigma_1$  with the imposed constraint  $a^2 + b^2 = 1$ . We find that  $K = a - ib(1-t/2)/\sqrt{1-t}$ , whereas the equation  $\delta\mathcal{L} = 0$  gives  $(\gamma/t)\sqrt{1-t}K + ib/(g+a) = 0$ . Eliminating in this equation  $a$  and  $b$ , one finally arrives after some algebra at the following saddle-point equation:

$$\frac{1}{t} = \frac{1}{2} - \frac{1}{\gamma K} + \frac{(K+g)/2}{\pm \sqrt{(K+g)^2 + 1 + 4/\gamma^2 - g^2 - 2/\gamma}}, \quad (7)$$

where the sign “+” must be taken. Independently on the taken choice of the sign, Eq. (7) can be further equivalently represented as the following fourth-order equation in  $K$ :

$$\begin{aligned} \frac{K^4 + 2gK^3}{t} \left(1 - \frac{1}{t}\right) + \frac{K^2}{4} \left[ \left(g + \frac{2}{\gamma}\right)^2 - \frac{8g}{t\gamma} - \left(1 - \frac{2}{t}\right)^2 \right] \\ + \frac{K}{\gamma} \left(1 - \frac{2}{t}\right) - \frac{1}{\gamma^2} = 0. \end{aligned} \quad (8)$$

In the zero absorption limit,  $\gamma \rightarrow 0$  with fixed  $t/\gamma = \tau$ , this equation simplifies to the cubic one found earlier [16].

The choice of the sign “+” is justified by two reasons. First, only then the complex solution of Eq. (7) yields the distribution  $\mathcal{P}(r)$  nonzero at positive  $r$ . Second, the function  $K(t)$ , as follows directly from its definition, must have at  $t \rightarrow -\infty$  the positive limit  $K_\infty = t_H^{-1} \text{tr} Q_\gamma < 1$  [8], which is the mean scaled proper delay time  $\langle \tau \rangle$ . The equation for the latter follows then readily from (8) as  $[(g+2/\gamma)^2 - 1]\langle \tau \rangle^2 + 4\langle \tau \rangle/\gamma - 4/\gamma^2 = 0$  which gives  $\langle \tau \rangle = T/(\gamma+T)$ , reproducing thus Eq. (3) by virtue of  $\langle r \rangle = 1 - \gamma\langle \tau \rangle$ .

The higher moments of the distribution can be easily calculated in the same way, exploiting the  $1/t$  expansion further. For example, substituting  $K = \langle \tau \rangle + \gamma\langle \tau^2 \rangle/t + O(t^{-2})$  in Eq. (8) and equating there the  $1/t$  term to zero, we find  $\langle \tau^2 \rangle = \langle \tau \rangle^2 [\gamma^2 + 2(\gamma+T)]/(\gamma+T)^2$ . That yields the variance of reflection eigenvalues as follows:

$$\langle r^2 \rangle - \langle r \rangle^2 = \frac{\gamma^2 T^2 (2\gamma + 2T(1-\gamma) - T^2)}{(\gamma+T)^4}. \quad (9)$$

Remarkably, the variance and, therefore, fluctuations of  $r$  are suppressed in the both limits of weak and strong absorption. This reflects the fact that  $R$  becomes the unit matrix as  $\gamma \rightarrow 0$  whereas  $R \rightarrow 1 - T$  (or  $S_\gamma \rightarrow \bar{S}$ ) as  $\gamma \rightarrow \infty$ , see also Eq. (3).

To understand qualitatively the structure of the solutions of the saddle-point equation, it is instructive to consider the behavior of  $1/t$  as a function of  $K$  shown schematically on Fig. 1. We readily see that the distribution function  $\mathcal{P}(r)$  is nonzero only in the finite domain  $r_{\min} < r < r_{\max}$ . The value of the upper border  $r_{\max} < 1$  at any  $g$  and  $\gamma$ . In a vicinity of  $r_{\max}$  the distribution behaves as  $\mathcal{P}(r) \propto \sqrt{r_{\max} - r}$ . For the lower border we find  $r_{\min} = 0$  and  $\mathcal{P}(r) \propto 1/\sqrt{r}$ , when  $(g - 2/\gamma)^2 \leq 1$  or  $T \leq \gamma \leq T/(1-T)$ , and  $r_{\min} > 0$  with  $\mathcal{P}(r) \propto \sqrt{r - r_{\min}}$  otherwise [21]. Such a behavior should be contrasted with that of the few-channel case [8] when all values of  $r$  from zero to one are permitted. Further analytical study is possible in the following particular cases, which we consider in detail now.

### PERFECT COUPLING, $g = T = 1$

In this case, the region between the two vertical dashed-dotted lines on Fig. 1 shrinks to the line  $K = -1$ . Eliminating the resulting common factor  $(K+1)$ , one gets from Eq. (8) the following cubic equation:

$$K^3 + K^2 + \frac{(1-r)^2}{r\gamma^2} \left[ K \left( \frac{2\gamma}{1-r} - 1 - \gamma \right) + 1 \right] = 0, \quad (10)$$

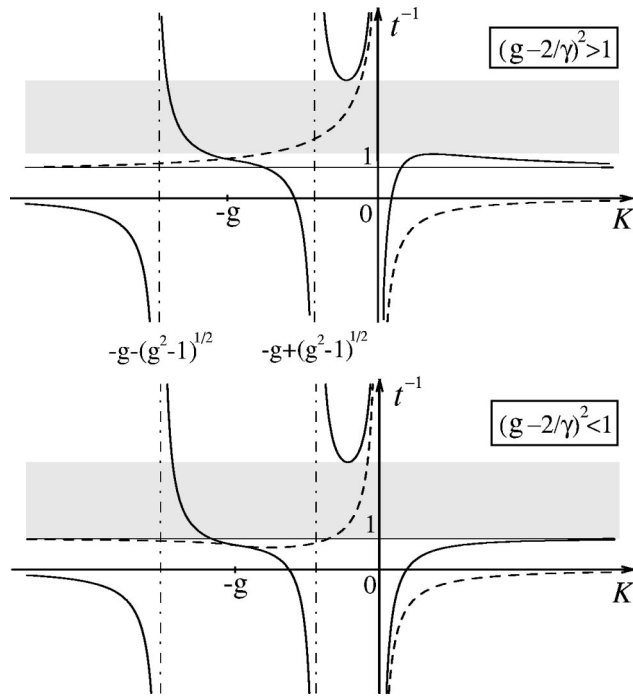


FIG. 1. The schematic plot of  $1/t$  as a function of  $K$ . The solid and dashed lines correspond to the choices of the “+” and “-” signs in Eq. (7), respectively. Altogether they form the complete set of solutions of the fourth-order equation (8). In the shadowed region the latter equation has two complex-conjugated roots (and two real roots irrelevant for our purposes), with  $t^{-1} > 1$  ( $r > 0$ ). Thus, only in this region the corresponding density is nonzero. Another solution with  $0 < t^{-1} < 1$  is unphysical.

written already in the variable  $r = 1 - t$ . We find that the discriminant  $D = [(1 + \gamma)/(3\gamma^2 r)]^3 (r_+ - r)(r - r_-)(1 - r)^4$  of this equation, with

$$r_{\pm} = \frac{8 + 20\gamma^2 - \gamma^4 \pm \gamma(8 + \gamma^2)^{3/2}}{8(1 + \gamma)^3}, \quad (11)$$

is positive at  $r > 0$  (and thereby  $\text{Im } K$  is nonzero) only in the domain  $\max[0, r_-] < r < r_+$ . This explicitly sets the exact borders for the searched distribution.  $\mathcal{P}(r)$  can be found from Eq. (10) by applying Cardan’s formulas, reproducing exactly the result of Ref. [5] obtained by a different method. When absorption is weak,  $\gamma < 1$ , the behavior of  $\mathcal{P}(r)$  can be well approximated by the following interpolation expression [22]:

$$\mathcal{P}_{\gamma < 1}(r) = \frac{C_{\gamma}}{2\pi} \frac{\sqrt{(r_+ - r)(r - r_-)}}{(1 - r)^2 \sqrt{r}}, \quad (12)$$

with  $C_{\gamma} = 1 + \frac{3}{2}\gamma + O(\gamma^2)$  being the normalization constant. In the opposite case of strong absorption, the distribution is found to be close to

$$\mathcal{P}_{\gamma > 1}(r) = \frac{2\sqrt{1 - r_+}}{\pi r_+} \frac{\sqrt{r_+ / r - 1}}{(1 - r)^2}. \quad (13)$$

The limiting distribution (12) or (13) becomes asymptotically exact as  $\gamma$  diminishes or grows, respectively. All these features are illustrated on Fig. 2(a).

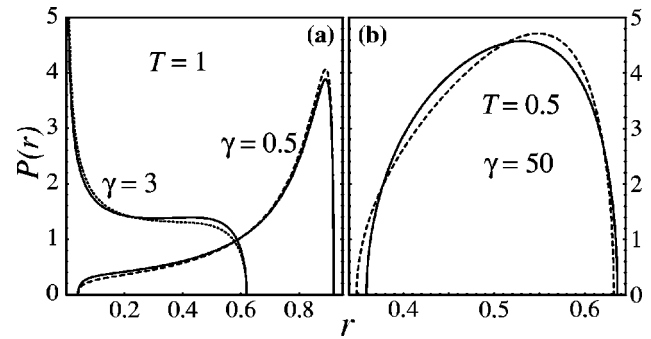


FIG. 2. The exact reflection distribution (solid lines) compared to the corresponding approximate expressions (12), (13), and (15) at perfect ( $T = 1$ , left) and nonperfect ( $T = 0.5$ , right) coupling.

### SPECIAL COUPLING, $g^2 = 1 + 4/\gamma^2$

Under this condition Eq. (7) reduces to a quadratic equation in  $K$ , giving readily

$$\mathcal{P}_g(r) = \frac{2\sqrt{1 - r_g}}{\pi r_g} \frac{\sqrt{r_g / r - 1}}{(1 - r)^2}, \quad (14)$$

with  $r_g = 2/(1 + \sqrt{1 + \gamma^2/4}) = 2/(1 + g/\sqrt{g^2 - 1})$ .

### WEAK COUPLING, $g \approx 2T \gg 1$

One can find approximate expressions for  $\mathcal{P}(r)$ , making use of the scaling  $K \sim O(1/g)$  in the exact equations. When absorption is weak, we may expand the square root in Eq. (7) and keep only the main contribution  $(2g/\gamma)(1 + 2gK)^{-1}$  instead of the last term there. That leads again to a quadratic equation in  $K$ , giving readily the behavior  $\mathcal{P}(r) \approx (\gamma/2g\pi^2)^{1/2} \sqrt{r_{\max} - r}/(1 - r)^2$  near the upper border  $r_{\max} \approx 1 - \gamma/8g$ . For  $\gamma$  from the small interval  $[T, T/(1 - T)]$ , when  $r_{\min} = 0$ ,  $\mathcal{P}(r)$  is given in the leading approximation by Eq. (14), with  $r_g$  replaced by  $r_{\max}$ , because the applicability condition for Eq. (14) is now satisfied up to  $O(g^{-2})$ . At other values of small  $\gamma$  there is no reliable approximation for the lower border available and  $r_{\min} > 0$  is to be found from the general equation [21]. At last, for very strong absorption and arbitrary nonperfect coupling,  $T \neq 1$ , we can use scaling  $K \sim O(1/\sqrt{\gamma})$  in Eq. (7). That allows us to replace the last term there with  $\frac{1}{2}(K + g)/(1 + gK - 2/\gamma)$ , yielding finally

$$\mathcal{P}_{\gamma \gg 1}(r) = \frac{C}{2\pi} \frac{\sqrt{(r_0 + \Delta r - r)(r - r_0 + \Delta r)}}{(1 - r)^2 (1 - T + r)}, \quad (15)$$

with the normalization constant  $C \approx \gamma - 2(1 - T)$  and

$$r_0 \approx 1 - T - \frac{2T(2 - 3T)}{\gamma} \quad \text{and} \quad \Delta r \approx 2\sqrt{2T} \sqrt{\frac{1 - T}{\gamma}},$$

which is valid when  $r_0 \gg \Delta r$  or  $\gamma \gg 8T^2/(1 - T)$ . Expression (15) becomes asymptotically exact as  $\gamma$  grows, approaching the “semicircle” distribution with the center at  $r_0$  and the radius  $\Delta r$ , see Fig. 2(b).

## THERMAL EMISSION

As an application of our results we consider thermal emission from random media. In his seminal paper Beenakker [5] has shown that the quantum optical problem of the photon statistics can be reduced to a computation of the  $S$  matrix of the classical wave equation. In particular, chaotic radiation may be characterized by the effective number  $\nu_{\text{eff}}$  degrees of freedom as follows:  $\nu_{\text{eff}}/\nu = (1 - \langle r \rangle)^2 / \langle (1 - r)^2 \rangle \leq 1$  [5], with  $\nu_{\text{eff}} = \nu$  for blackbody radiation [23]. We find using Eq. (9) that at any  $\gamma$  and  $T$

$$\frac{\nu_{\text{eff}}}{\nu} = \frac{\langle \tau \rangle^2}{\langle \tau^2 \rangle} = \frac{(\gamma + T)^2}{\gamma^2 + 2(\gamma + T)}, \quad (16)$$

and a mean photocount  $\bar{n} = \nu f \gamma T / (\gamma + T)$ , with  $f$  being a Bose-Einstein function. The earlier result [5] is reproduced at  $T = 1$ . Upon the substitution of  $\gamma$  with  $-\gamma$  one can use [5]  $\bar{n}$  and  $\nu_{\text{eff}}/\nu$  (16) even for amplified spontaneous emission below the laser threshold,  $\gamma < T$ . In this case our mean photocount agrees with the findings of Ref. [24], where the general theory of photocount statistics in random amplifying media was developed. In the limit of vanishing absorption

or amplification, the ratio  $\nu_{\text{eff}}/\nu$  is  $T/2$ . The large  $\gamma$  expansion  $\nu_{\text{eff}}/\nu = 1 - 2(1 - T)/\gamma + O(\gamma^{-2})$  shows that the saturation to the blackbody limit gets slower when transmission  $T < 1$ .

In conclusion, for many-channel chaotic systems we have derived the general distribution of reflection eigenvalues at arbitrary values of absorption and transmission. We note that due to a duality relation [5,25], an amplifying system in the linear regime ( $\Gamma_a < \Gamma_w$ ) is directly linked to the dual absorbing one through the change of the sign of  $\Gamma_a$  in (1) and correspondingly thereafter. As a result, the reflection matrices (and their eigenvalues) of dual systems are each other's reciprocal. Therefore, the analysis presented can straightforwardly be extended to the case of linear amplification that might be also relevant for the rapidly developing field of random lasers [4,5,24–26].

## ACKNOWLEDGMENTS

We are grateful to G. Hackenbroich and C. Viviescas for useful discussions. The financial support by the SFB/TR 12 der DFG is acknowledged with thanks.

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- [22] The square root in the denominator here  $\sqrt{r} \approx \frac{1}{2}(1+r)$  for  $r$  close to 1, which could be obtained from a quadratic approximation of Eq. (10) when  $\gamma \ll 1$ . It is numerically found that  $\sqrt{r}$  gives the correct tendency at moderate  $\gamma \approx 1$ , i.e., an enhancement of  $\mathcal{P}(r)$  at smaller  $r$ .
- [23] Registration of  $n$  photons in the frequency window  $\delta\omega$  during the large time  $t \gg 1/\delta\omega$  yields the negative-bimodal distribution of photocounts with  $\nu = Mt\delta\omega/2\pi$  degrees of freedom; see L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge University Press, Cambridge, UK, 1995).
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